SOLUTIONS
UBC Math104/184 Exam (December 2006)

1. (a) $\lim_{x\to-2} f(x) = \lim_{x\to-2}\frac{x^2-3x-10}{x+2} = \lim_{x\to-2}\frac{(x-5)(x+2)}{x+2} = \lim_{x\to-2}(x-5) = -7$.

If $f(x)$ is to be continuous at $x=-2$, then we need $\lim_{x\to-2} f(x) = f(-2)$. Since $f(-2)=c$, then $c=-7$.

(b) $$\lim_{t\to0}\frac{\sqrt{t+9}-3}{\sqrt{t}} = \lim_{t\to0}\frac{\sqrt{t+9}-3}{\sqrt{t}}\cdot\frac{\sqrt{t+9}+3}{\sqrt{t+9}+3} = \lim_{t\to0}\frac{(t+9)-9}{\sqrt{t}\,(\sqrt{t+9}+3)} = \lim_{t\to0}\frac{t}{\sqrt{t}\,(\sqrt{t+9}+3)}$$
$$= \lim_{t\to0}\frac{\sqrt{t}}{\sqrt{t+9}+3} = \frac{0}{\sqrt{9}+3} = 0$$

(c) $$\lim_{h\to0}\frac{(h+3)^2-9}{(h-5)^2-25} = \lim_{h\to0}\frac{(h^2+6h+9)-9}{(h^2-10h+25)-25} = \lim_{h\to0}\frac{h^2+6h}{h^2-10h} = \lim_{h\to0}\frac{h(h+6)}{h(h-10)} = \lim_{h\to0}\frac{h+6}{h-10} = \frac{6}{-10} = -\frac{3}{5}$$

(d) $A = Pe^{rt}$, where $r = 6\% = 0.06$, and P is the principal (the original amount). When the amount has tripled, $A=3P$, so
$$3P = Pe^{0.06t} \;\Rightarrow\; 3 = e^{0.06t} \;\Rightarrow\; \ln 3 = 0.06t \;\Rightarrow\; t = \frac{\ln 3}{0.06} \approx 18.31 \text{ years}.$$

(e) $y' = \frac{1}{4}\cdot 2(2x+1)\cdot 2 = 2x+1$. When the tangent line is parallel to the line $y-3x=1$ (or $y=3x+1$), its slope ($y'=2x+1$) is the same as the slope of the line $y=3x+1$ (namely $m=3$), so $2x+1=3$ or $2x=2$ or $x=1$. Therefore $y=\frac{1}{4}(2x+1)^2=\frac{1}{4}(2+1)^2=\frac{9}{4}$. So the point is $(1,\frac{9}{4})$.

(f) $y' = e^{3x}\cdot 3 + e^{-2x}\cdot(-2) = 3e^{3x}-2e^{-2x}$. The tangent line has slope zero when
$$3e^{3x} = 2e^{-2x} \;\Rightarrow\; \frac{e^{3x}}{e^{-2x}} = \frac{2}{3} \;\Rightarrow\; e^{5x} = \frac{2}{3} \;\Rightarrow\; 5x = \ln\tfrac{2}{3} \;\Rightarrow\; x = \tfrac{1}{5}\ln\tfrac{2}{3}.$$

(g) $f'(x) = -2(\sin^{-1}x)^{-3}\cdot\frac{1}{\sqrt{1-x^2}} = -\frac{2}{(\sin^{-1}x)^3}\cdot\frac{1}{\sqrt{1-x^2}}$. Therefore
$$f'(\tfrac{1}{\sqrt2}) = -\frac{2}{(\sin^{-1}\frac{1}{\sqrt2})^3}\cdot\frac{1}{\sqrt{1-(\frac{1}{\sqrt2})^2}} = -\frac{2}{(\frac{\pi}{4})^3}\cdot\frac{1}{\sqrt{1-\frac12}} = -\frac{2}{\frac{\pi^3}{64}}\cdot\frac{1}{\sqrt{\frac12}} = -2\cdot\frac{64}{\pi^3}\cdot\sqrt2 = -\frac{128\sqrt2}{\pi^3}.$$

(h) $g(x)=\sqrt{1+3f(x)}$, so $g(0)=\sqrt{1+3f(0)}=\sqrt{1+3\cdot1}=\sqrt4=2$. Therefore the point on the graph of $y=g(x)=\sqrt{1+3f(x)}$ is $(0,2)$.
$$g'(x) = \tfrac12(1+3f(x))^{-1/2}\cdot 3f'(x) = \frac{3f'(x)}{2\sqrt{1+3f(x)}}.$$
Therefore, the slope of the tangent line at the point $(0,2)$ is
$$m = g'(0) = \frac{3f'(0)}{2\sqrt{1+3f(0)}} = \frac{3\cdot4}{2\sqrt{1+3\cdot1}} = \frac{12}{2\sqrt4} = 3.$$
So the equation of the tangent line is $y=mx+b=3x+2$ (since $(0,2)$ is the y-intercept).

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(i) $\frac{d}{dx}(y+\ln(y+3))=\frac{d}{dx}(x^2) \Rightarrow \frac{dy}{dx}+\frac{1}{y+3}\cdot\frac{dy}{dx}=2x \Rightarrow (y+3)\frac{dy}{dx}+1\cdot\frac{dy}{dx}=2x(y+3)$

$$\Rightarrow (y+4)\frac{dy}{dx}=2x(y+3) \Rightarrow \frac{dy}{dx}=\frac{2x(y+3)}{y+4}.$$

(j) $f'(x)=\dfrac{(x+1)\cdot\left(0+\frac{1}{x+1}\cdot 1\right)-(1+\ln(x+1))\cdot 1}{(x+1)^2} = \dfrac{1-1-\ln(x+1)}{(x+1)^2} = -\dfrac{\ln(x+1)}{(x+1)^2}.$

$f(x)$ is increasing when $f'(x)>0$, which occurs when $\ln(x+1)<0$ (since $(x+1)^2$ is always positive), i.e. when $x+1<e^0=1$ or $x<0$. So $f(x)$ is increasing on the interval $(-1,0)$. (Note that $f(x)$ is not defined when $x\le -1$.)

(k) $f'(x)=\dfrac{(x+2)\cdot 1-x\cdot 1}{(x+2)^2}=\dfrac{2}{(x+2)^2}=2(x+2)^{-2}$ so $f''(x)=-4(x+2)^{-3}\cdot 1=-\dfrac{4}{(x+2)^3}.$

Therefore $f(x)=\dfrac{x}{x+2}$ is concave down when $f''(x)<0$, which occurs when $(x+2)^3>0$ (since -4 is always negative), i.e. when $x+2>0$ or $x>-2$. So $f(x)$ is concave down on the interval $(-2,\infty)$.

(l) $f'(x)=\dfrac{(x^2+4)\cdot 1-x\cdot 2x}{(x^2+4)^2}=\dfrac{x^2+4-2x^2}{(x^2+4)^2}=\dfrac{4-x^2}{(x^2+4)^2}.$

At a critical point $f'(x)=0$ so $4-x^2=0$ or $x^2=4$ and therefore $x=\pm 2$. So there is only one critical point inside the interval $[-1,5]$, namely $x=2$. There are also the two endpoints $x=-1$ and $x=5$ to consider.

$f(2)=\dfrac{2}{2^2+4}=\dfrac{2}{8}=\dfrac{1}{4}=0.25$; $\quad f(-1)=\dfrac{-1}{(-1)^2+4}=-\dfrac{1}{5}=-0.20$; $\quad f(5)=\dfrac{5}{5^2+4}=\dfrac{5}{29}\approx 0.17$.

The global minimum occurs when $x=-1$ and has the value $f(-1)=-\frac{1}{5}$ so it is at the point $(-1,-\frac{1}{5})$.

(m) This is a geometric series of the form $a+ar+ar^2+ar^3+\cdots$ where $a=-\frac{4}{5}$ and $r=-\frac{1}{5}$. The sum is therefore

$$a+ar+ar^2+ar^3+\cdots=\frac{a}{1-r}=\frac{-\frac{4}{5}}{1+\frac{1}{5}}=\frac{-\frac{4}{5}}{\frac{6}{5}}=-\frac{2}{3}.$$

(n) Since $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$, therefore

$$e^{-x}=1+(-x)+\frac{(-x)^2}{2!}+\frac{(-x)^3}{3!}+\frac{(-x)^4}{4!}+\cdots=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\cdots.$$

So $$(x+1)e^{-x} = xe^{-x} + e^{-x} = x\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)$$

$$= (x - x^2 + \tfrac{1}{2}x^3 - \tfrac{1}{6}x^4 + \tfrac{1}{24}x^5 + \cdots) + (1 - x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \cdots)$$

$$= 1 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{8}x^4 + \cdots.$$

So $c_2 = -\frac{1}{2}$, since that is the coefficient of x^2.

2. $$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\left[\frac{x+h}{1-3(x+h)} - \frac{x}{1-3x}\right] \div h = \lim_{h\to 0}\left[\frac{x+h}{1-3x-3h} - \frac{x}{1-3x}\right]\cdot\frac{1}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)(1-3x) - x(1-3x-3h)}{(1-3x)(1-3x-3h)}\cdot\frac{1}{h} = \lim_{h\to 0}\frac{x + h - 3x^2 - 3xh - x + 3x^2 + 3xh}{h(1-3x)(1-3x-3h)}$$

$$= \lim_{h\to 0}\frac{h}{h(1-3x)(1-3x-3h)} = \lim_{h\to 0}\frac{1}{(1-3x)(1-3x-3h)} = \frac{1}{(1-3x)^2}.$$

3. Since demand q is a linear function of the price p, $q = ap + b$ where a and b are constants.

When $p = 7$, $q = 60$, so $60 = a\cdot 7 + b$, and when $p = 5$, $q = 66$, so $66 = a\cdot 5 + b$.

Therefore $60 - 66 = (7a + b) - (5a + b)$, so $-6 = 2a$ and $a = -3$.

Then $b = 60 - 7a = 60 - (-21) = 81$. So $q = ap + b = -3p + 81 = 81 - 3p$.

The profit is given by

$$P = R - C = pq - 3q = (p-3)q = (p-3)(81-3p) = 81p - 243 - 3p^2 + 9p = 90p - 243 - 3p^2.$$

Therefore $\frac{dP}{dp} = 90 - 6p = 0$ at a critical point, so $6p = 90$ and $p = 15$.

It should charge \$15 for the appetizer.

4. (a) $$f'(x) = \frac{(x^2+1)\cdot 2 - 2x\cdot 2x}{(x^2+1)^2} = \frac{2x^2 + 2 - 4x^2}{(x^2+1)^2} = \frac{2 - 2x^2}{(x^2+1)^2} = \frac{2(1-x^2)}{(x^2+1)^2}.$$

At a critical point $f'(x) = 0$, so $1 - x^2 = 0$. Therefore $x^2 = 1$, so $x = \pm 1$.

So there are two critical points, $(1, f(1)) = (1,1)$ and $(-1, f(-1)) = (-1,-1)$.

$f(x)$ is increasing if $f'(x) = \frac{2(1-x^2)}{(x^2+1)^2} > 0$, i.e. if $x^2 < 1$ or $-1 < x < 1$. So $f(x)$ is increasing on the interval $(-1,1)$ (and therefore decreasing on the intervals $(-\infty,-1)$ and $(1,\infty)$).

(b) $f''(x) = \dfrac{(x^2+1)^2(0-4x)-(2-2x^2)\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4} = \dfrac{(x^2+1)(-4x)-(2-2x^2)\cdot 2\cdot 2x}{(x^2+1)^3}$

$$= \frac{(-4x)\,[(x^2+1)+(2-2x^2)]}{(x^2+1)^3} = \frac{-4x(3-x^2)}{(x^2+1)^3} = \frac{4x(x^2-3)}{(x^2+1)^3}.$$

At an inflection point $f''(x)=0$, so $4x(x^2-3)=0$. Therefore $4x=0$ or $x^2-3=0$, so $x=0$ or $x^2=3$. So there are three possible inflection points, $x=0$, $x=-\sqrt{3}$ and $x=\sqrt{3}$.

When $x<-\sqrt{3}$, $f''(x)=\dfrac{4x(x^2-3)}{(x^2+1)^3}=\dfrac{(\text{negative})(\text{positive})}{\text{positive}}=\text{negative}$, so $f(x)$ is concave down.

When $-\sqrt{3}<x<0$, $f''(x)=\dfrac{4x(x^2-3)}{(x^2+1)^3}=\dfrac{(\text{negative})(\text{negative})}{\text{positive}}=\text{positive}$, so $f(x)$ is concave up.

When $0<x<\sqrt{3}$, $f''(x)=\dfrac{4x(x^2-3)}{(x^2+1)^3}=\dfrac{(\text{positive})(\text{negative})}{\text{positive}}=\text{negative}$, so $f(x)$ is concave down.

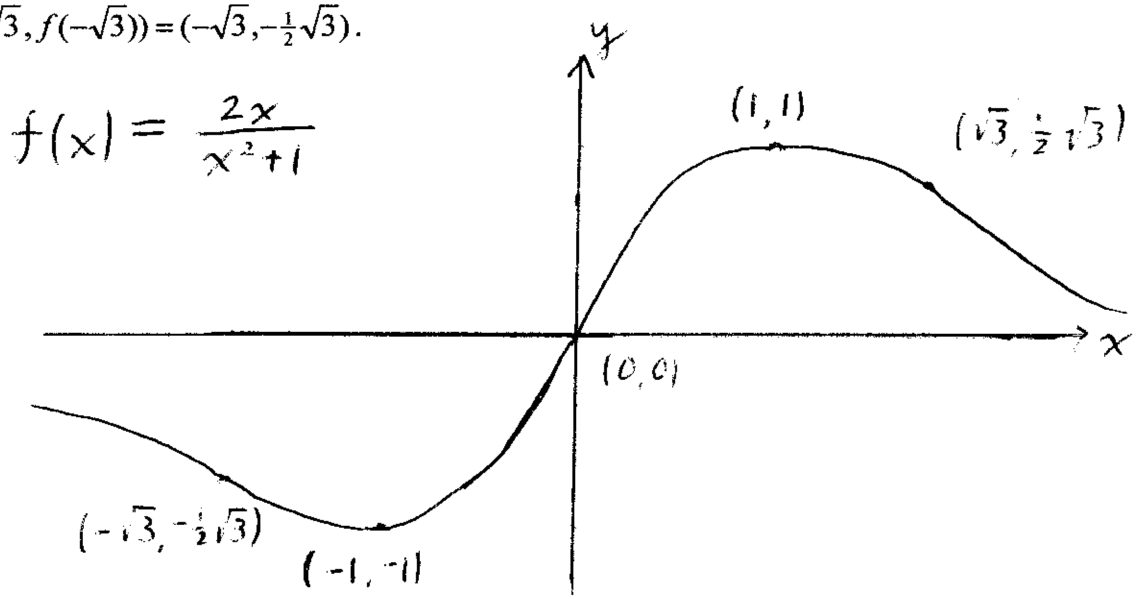
When $x>\sqrt{3}$, $f''(x)=\dfrac{4x(x^2-3)}{(x^2+1)^3}=\dfrac{(\text{positive})(\text{positive})}{\text{positive}}=\text{positive}$, so $f(x)$ is concave up.

Thus $f(x)$ is concave down on the intervals $(-\infty,-\sqrt{3})$ and $(0,\sqrt{3})$ and concave up on the intervals $(-\sqrt{3},0)$ and $(\sqrt{3},\infty)$.

(c) The critical point $(1,1)$ is a local maximum since $f''(1)=\dfrac{4\cdot(-2)}{(1+1)^3}<0$.

The critical point $(-1,-1)$ is a local minimum since $f''(1)=\dfrac{(-4)\cdot(-2)}{(1+1)^3}>0$.

Since the concavity changes at each of the three possible inflection points $x=0$, $x=\sqrt{3}$, and $x=-\sqrt{3}$, they are actual inflection points: $(0,f(0))=(0,0)$, $(\sqrt{3},f(\sqrt{3}))=(\sqrt{3},\frac{1}{2}\sqrt{3})$ and $(-\sqrt{3},f(-\sqrt{3}))=(-\sqrt{3},-\frac{1}{2}\sqrt{3})$.



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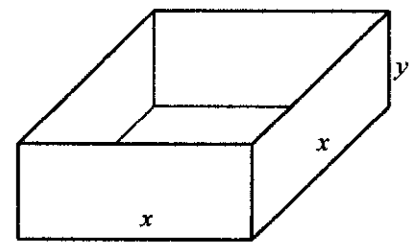

5. Let x be the length of the base of the box and y its height. The volume is $V = x^2 y$. The total cost is $C = 5x^2 + 4\cdot(2xy) = 5x^2 + 8xy$. Since the total cost is \$60, therefore $5x^2 + 8xy = 60$, so $8xy = 60 - 5x^2$. Thus $y = \dfrac{60-5x^2}{8x}$ so the volume is given by

$$V = x^2 y \;=\; x^2 \cdot \frac{60-5x^2}{8x} \;=\; \tfrac{1}{8}x\cdot(60-5x^2) \;=\; \tfrac{1}{8}(60x - 5x^3).$$

Therefore $\dfrac{dV}{dx} = \tfrac{1}{8}(60-15x^2) = \tfrac{15}{8}(4-x^2) = 0$ at a critical point. So $4 - x^2 = 0$, or $x^2 = 4$. Then $x = 2$ (since x must be positive). Therefore $y = \dfrac{60-5x^2}{8x} = \dfrac{60-5\cdot 2^2}{8\cdot 2} = \dfrac{60-20}{16} = \dfrac{5}{2}$. So $x = 2$ and $y = \frac{5}{2}$.

6. Plugging in $p = 30$ gives

$$30^2 + 2q^2 = 1100 \;\Rightarrow\; 2q^2 = 1100 - 30^2 = 1100 - 900 = 200 \;\Rightarrow\; q^2 = 100 \;\Rightarrow\; q = 10.$$

$$\frac{d}{dt}(p^2 + 2q^2) = \frac{d}{dt}(1100) \;\Rightarrow\; 2p\frac{dp}{dt} + 4q\frac{dq}{dt} = 0 \;\Rightarrow\; p\frac{dp}{dt} + 2q\frac{dq}{dt} = 0$$

Plugging in $p = 30$, $q = 10$ and $\dfrac{dp}{dt} = 2$ gives

$$30\cdot 2 + 2\cdot 10\frac{dq}{dt} = 0 \;\Rightarrow\; 60 + 20\frac{dq}{dt} = 0 \;\Rightarrow\; \frac{dq}{dt} = -\frac{60}{20} = -3.$$

Since revenue is given by $R = pq$,

$$\frac{dR}{dt} \;=\; p\frac{dq}{dt} + q\frac{dp}{dt} \;=\; 30\cdot(-3) + 10\cdot 2 \;= -90 + 20 \;= -70.$$

Revenue is decreasing at a rate of \$70 per month.